Catch them if you can!

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Course:Algorithms on Graphs

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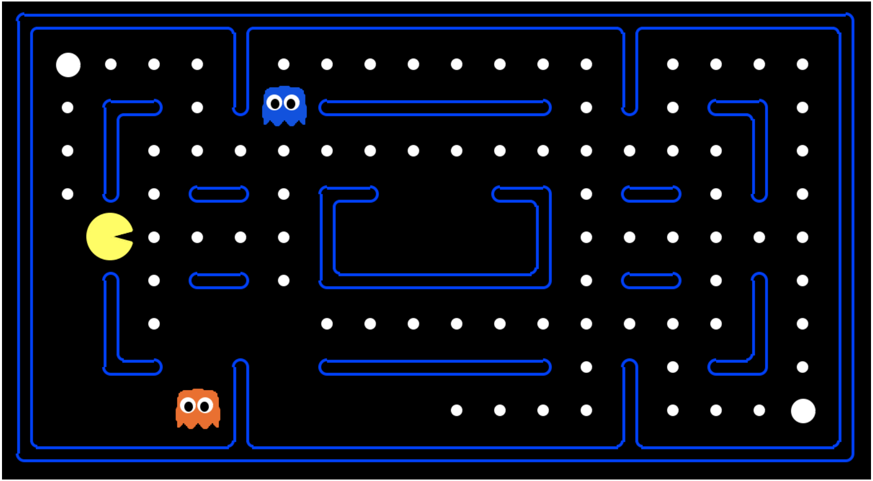
Thanks, Harmender, for introducing me to the problem and giving me your slides!

PacMan



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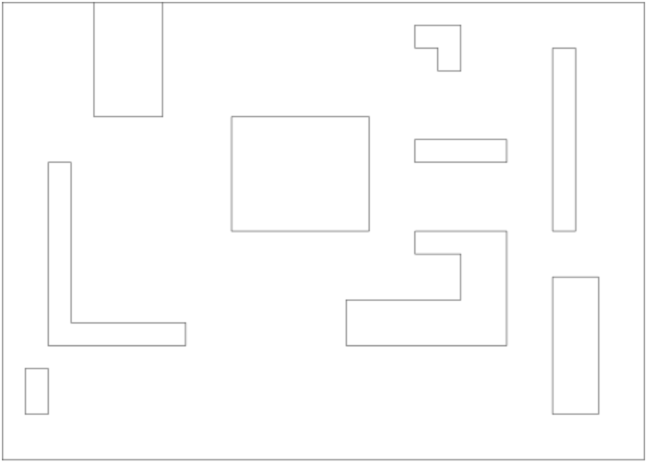

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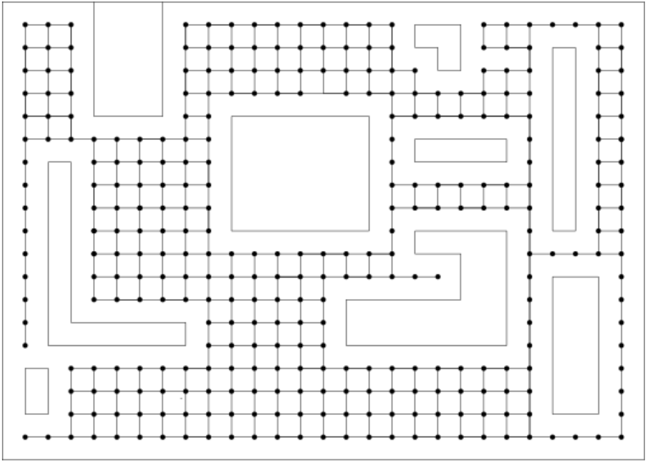

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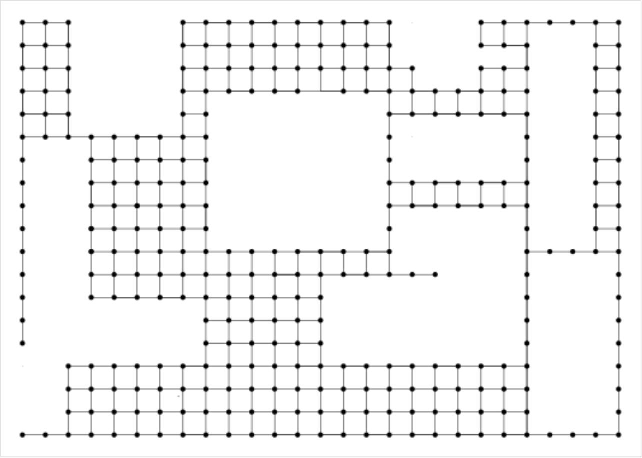

Playground

A simple, undirected, connected, and finite graph.

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Players

Two players: 1 cop and 1 robber.

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  For a vertex v, its closed neighborhood is $\{v\}$
  $\cup\{u : uv$ is an edge in $G\}$.

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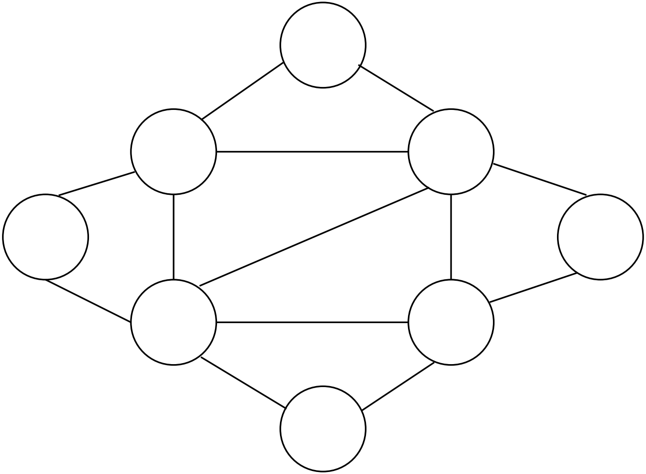
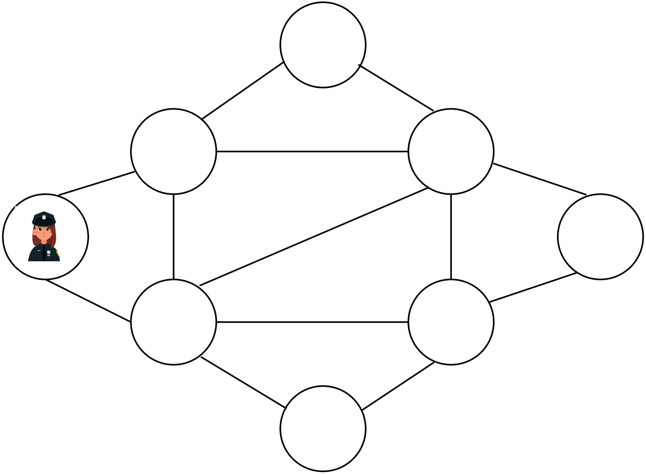
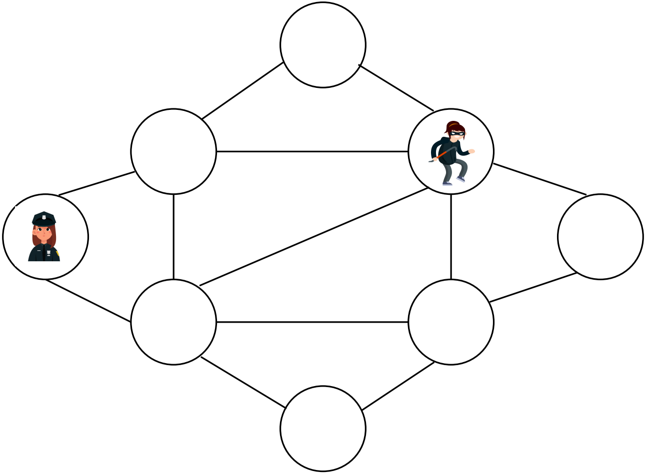
- Robber wins if it can evade the cop indefinitely.

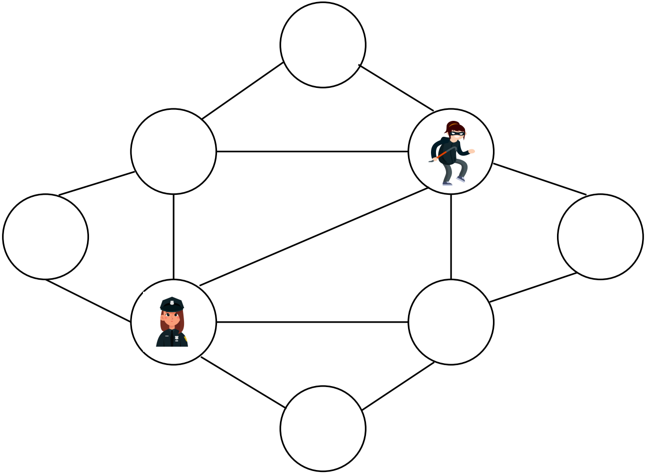




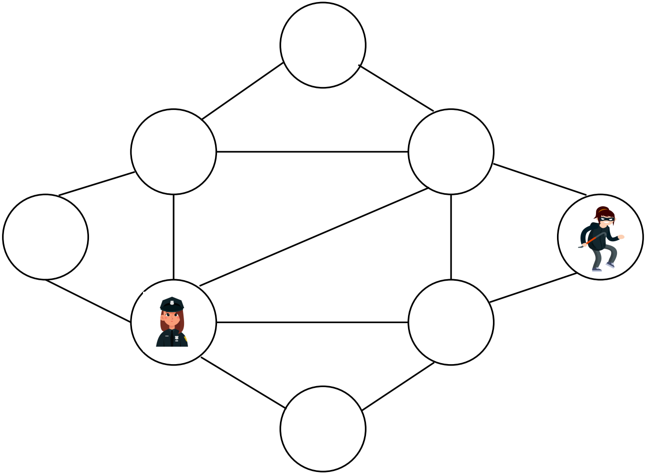
Cop enters the graph



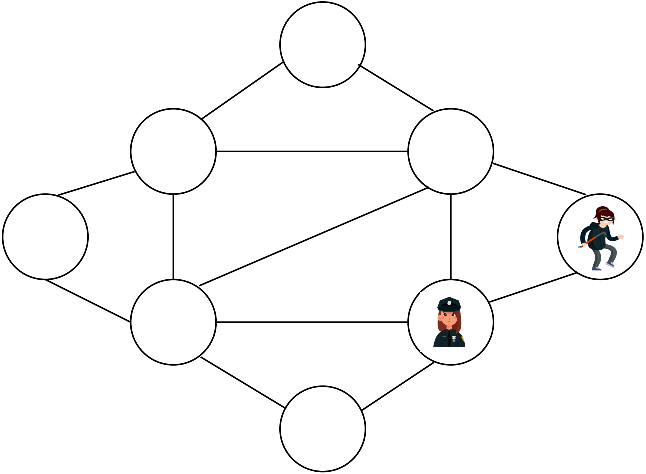
Robber enters the graph



Cop moves



Robber moves



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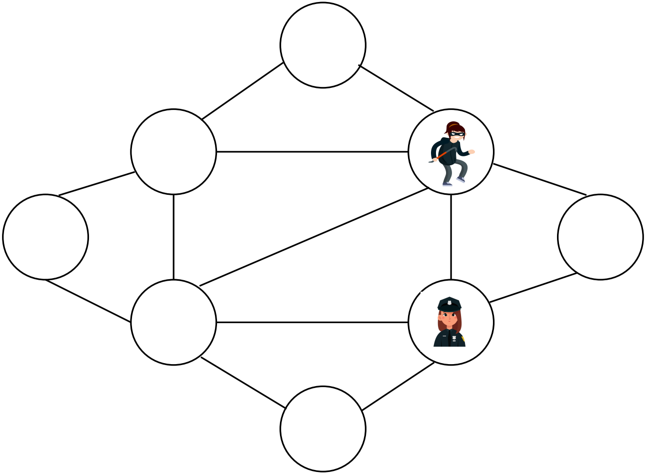
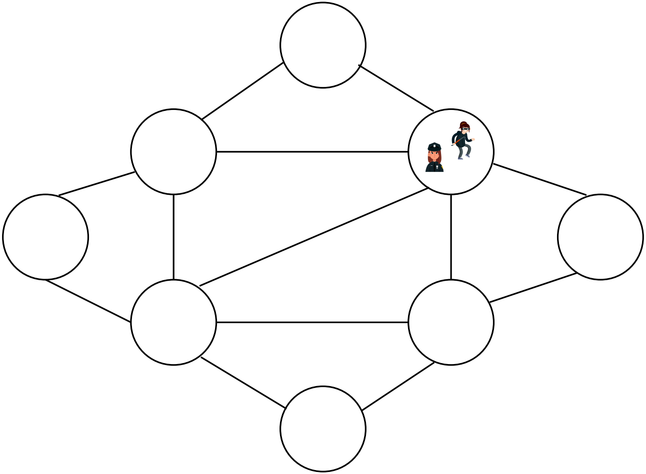


Robber moves



CAPTURE!

Cop-win graph

If the cop has a winning strategy!

Cop-win graph

If the cop has a winning strategy!
It can always capture the robber, no matter how the robber chooses to move.

Robber-win graph

If the robber can evade the cop forever, no matter how the cop chooses to move!

Start vertex is a part of the strategy!

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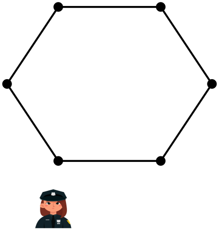

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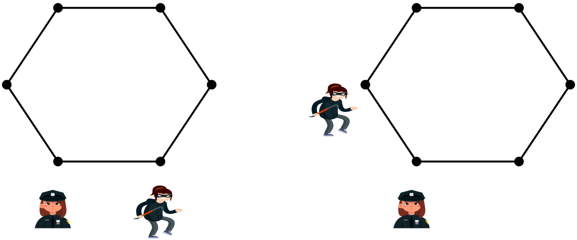

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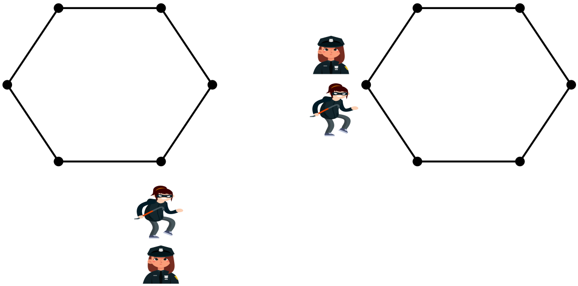
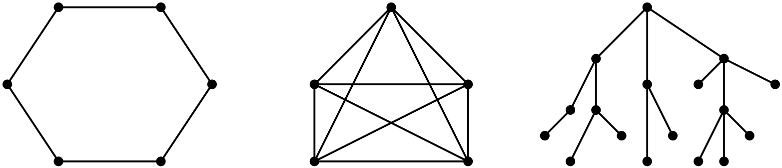
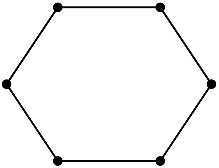
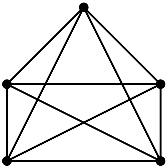


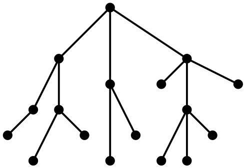




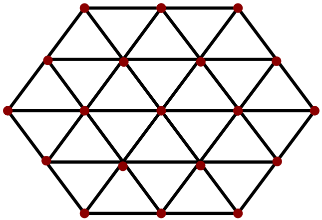
robber-win



cop-win



cop-win



What can you say about this graph?

When is a robber sure to be captured?

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The graph should have a Pitfall!

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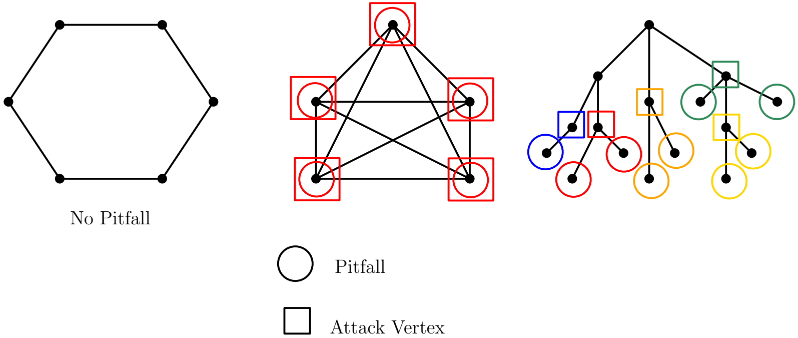
A pitfall is a vertex whose closed neighborhood is entirely covered by another vertex, called the attack vertex.

When is a robber sure to be captured?

The graph should have a Pitfall!

A pitfall is a vertex whose closed neighborhood is entirely covered by another vertex, called the attack vertex.

Definition: A pair of vertices (p, a) is considered a *pitfall* together with its *attack vertex* if $N(p) \cup \{p\} \subseteq N(a)$.


No Pitfall
Pitfall
Attack Vertex

Lemma: For a graph to be cop-win, it has to contain a pitfall.

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Is the converse true?

Theorem: Adding a pitfall does not change the winner!

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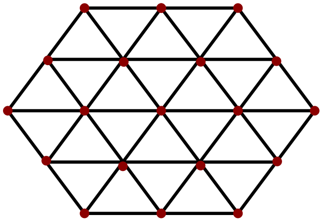
Corollary: Removing a pitfall does not change the winner!

Characterization of cop-win graphs

Theorem: G is a cop-win graph iff by successively removing pitfalls (in any order), G can be reduced to a single vertex.

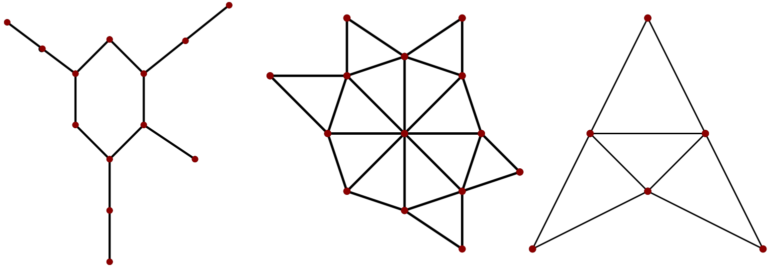
Characterization of cop-win graphs

Theorem: G is a cop-win graph iff by successively removing pitfalls (in any order), G can be reduced to a single vertex. Otherwise, the graph is robber-win.



What can you say about this graph now?

Cop-win or Robber-win?



In case the graph is a robber-win graph, what is the minimum number of cops required to guarantee the capture of the robber?

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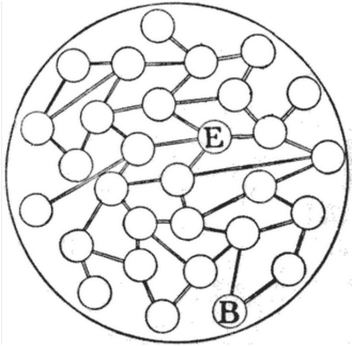
Cop number of the graph denoted by c(G)!

Some History

- Quillot in his Ph.D. thesis (1978).
- Independently by Nowakowski and Winkler (1983).
- Cop number introduced by Aigner and Fromme (1984).
- A detailed survey including some variants: Bonato and Nowakiwski (2011).

Prehistory

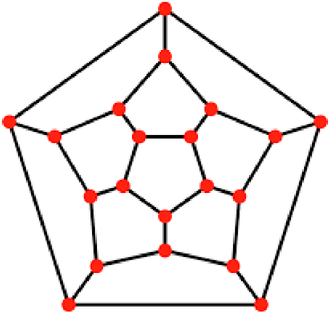
In the book *Amusements in Mathematics*, published in 1917, Henry Ernest Dudeney asked the following question.



How many cops are needed to capture the robber in a cycle?

How many cops are needed to capture the robber in a cycle?
Why? Explain your strategy!

What about this graph?



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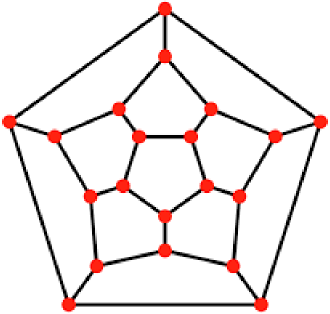
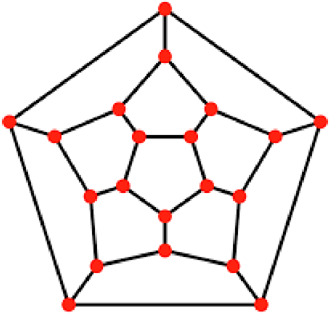


Note that it has no pitfall, so cop(dodecahedron) ≥ 2.

What about this graph?



Note that it has no pitfall, so cop(dodecahedron) ≥ 2.
Also, cop(dodecahedron) ≤ 20.

Are 2 cops sufficient?

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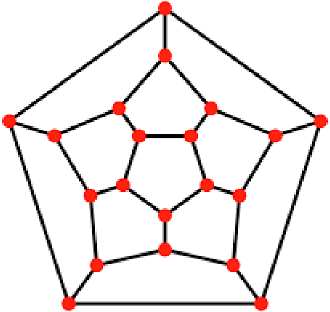
  NO because of the upcoming theorem.

Theorem: Let G be a graph with minimum degree at least d which contains no 3-cycles or 4-cycles. Then $c(G) \geq d$.



Thus, $3 \leq \text{cop}(\text{dodecahedron}) \leq 20$.

How about 10 cops?

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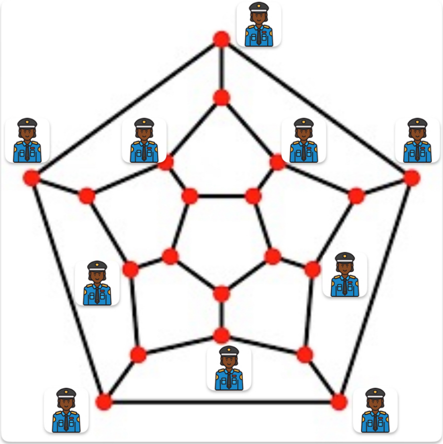
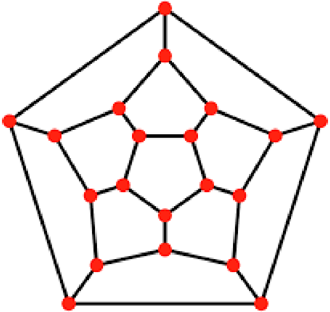




Thus, $3 \leq \text{cop}(\text{dodecahedron}) \leq 10$.

Are 3 cops sufficient?

Are 3 cops sufficient?

   Yes!

Thanks, Aigner and Fromme.

Theorem

For any planar graph G, $cop(G) \leq 3$.

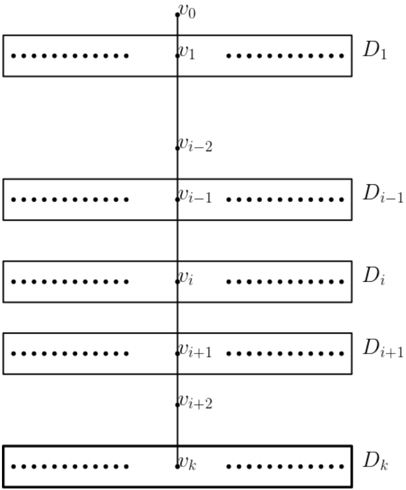
Guarding a shortest path

Guarding a shortest path

Lemma

Let G be any graph, and $P = \{u = v_0, v_1, v_2, \ldots, v_k = v\}$ be a shortest path between any two vertices u and v.
Then, a single cop C on P can, after a finite number of moves, prevent the robber R from entering P (that is, R will be immediately caught if he moves onto P).

Guarding a shortest path



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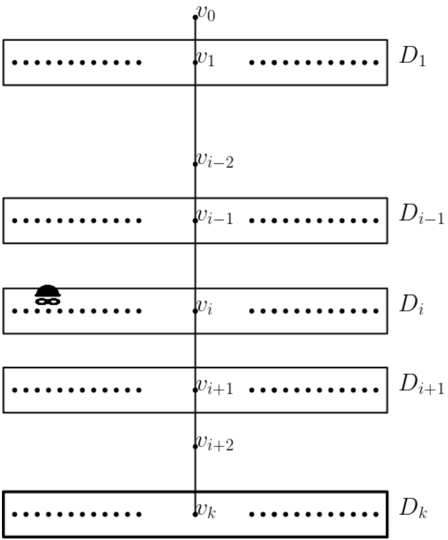

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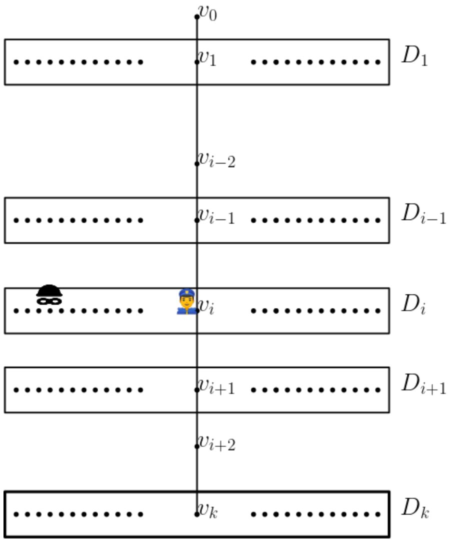

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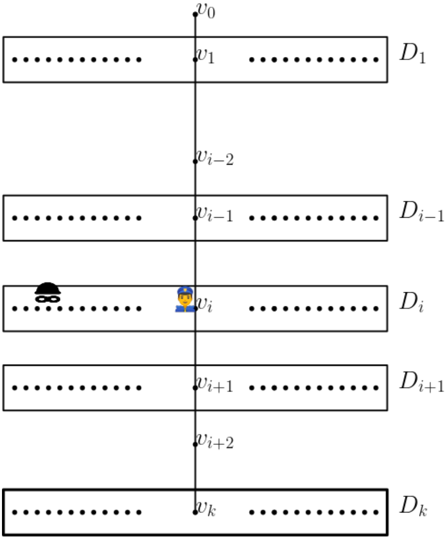
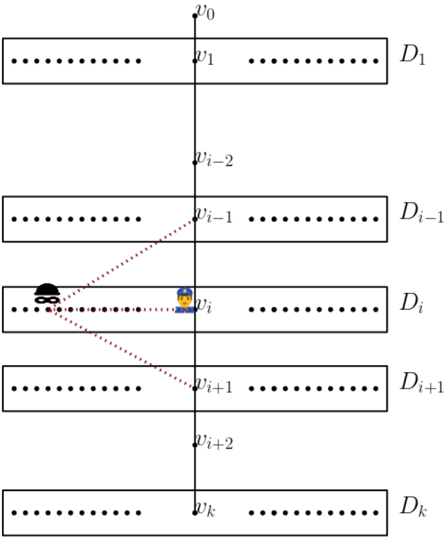


$$d(r,z) \geq d(c,z) \quad \text{for all} \quad z \in V(P) \qquad (\star)$$

Guarding a shortest path



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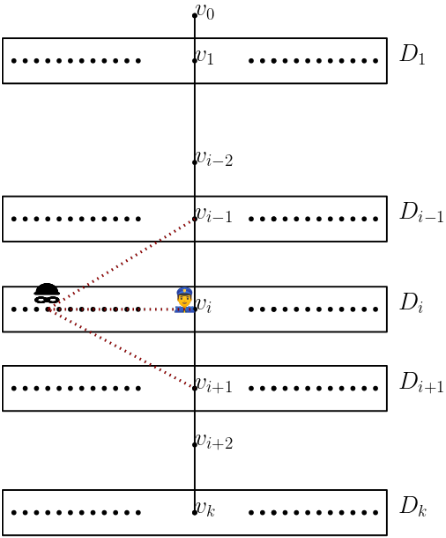

Thus, $d(r, z) \geq d(c, z)$ for all $z \in V(P)$ holds throughout.

Brief idea!

- Assign at each stage i to R a certain subgraph R_i called the Robber Territory which contains all vertices which R may still "safely" enter.

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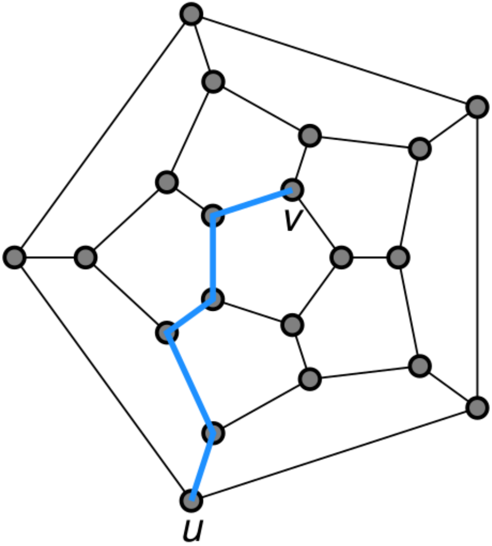
- Assign at each stage i to R a certain subgraph R_i called the Robber Territory which contains all vertices which R may still "safely" enter.
- After a finite number of cop-moves, R_i is reduced to $R_{i+1} \subset R_i$.
- Eventually, there is no vertex left for the robber to go.

Cops and Robber on Planar Graphs



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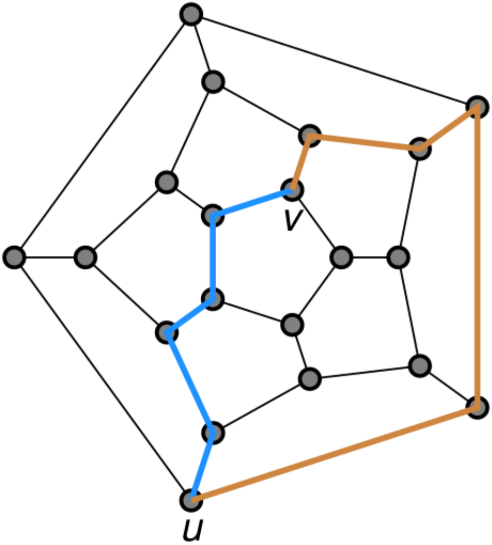

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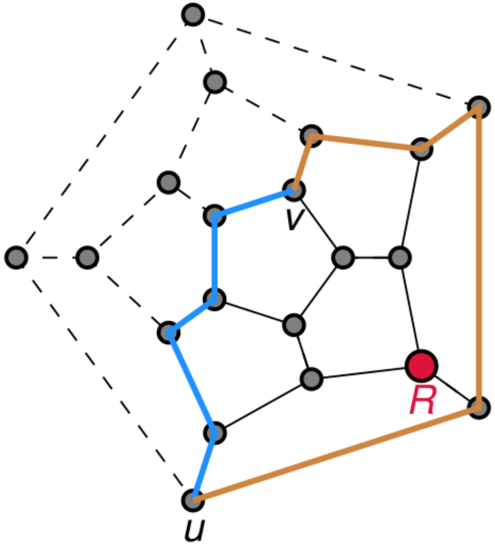

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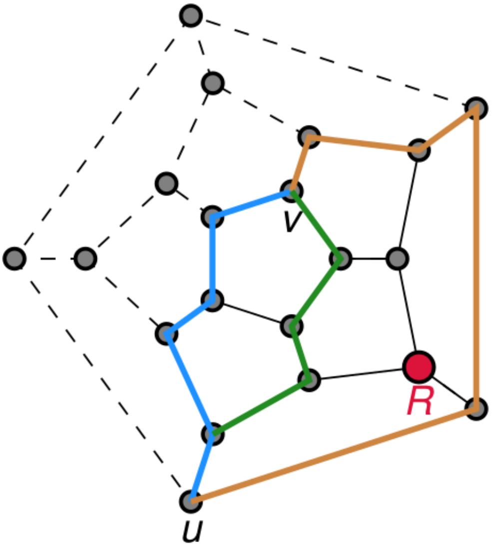

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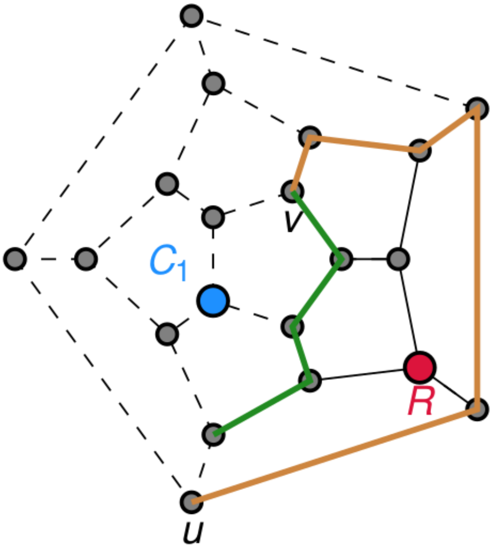

Are there graphs with unbounded cop number?

Theorem: To every $k \in \mathbb{N}$ there exists an k-regular graph without 3- or 4-cycles. Hence, for every k, there exists a graph G with $c(G) \geq k$.

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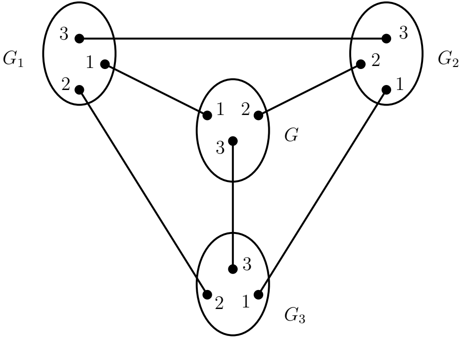
Proof:

- For $k = 1$, K_2 works.
- For $k = 2$, the 5-cycle C_5 works. Note that C_5 is 3-colorable.
- Assume, by induction, that we have an k-regular, 3-colorable graph G without 3- or 4-cycles.
- Create 3 copies of G, denoted G_1, G_2, G_3, and color them with 3 colors in the same way.
- Construct a new $k+1$-regular graph by:
  - Joining each vertex in G_1 to the corresponding vertex in G_2 if it is colored 3.
  - Joining each vertex in G_2 to the corresponding vertex in G_3 if it is colored 1.
  - Joining each vertex in G_3 to the corresponding vertex in G_1 if it is colored 2.

- After joining, interchange the colors:
  - Swap colors 3 and 1 in G_1.
  - Swap colors 2 and 1 in G_2.
  - Swap colors 3 and 2 in G_3.
- The resulting graph is $k+1$-regular, without 3- or 4-cycles, and remains 3-colorable.



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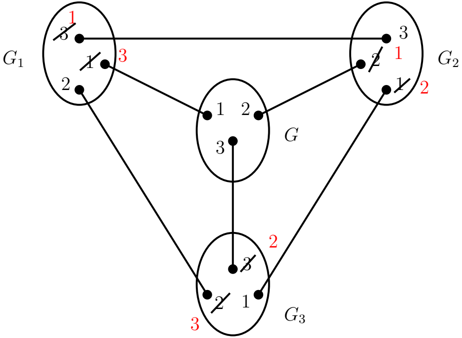

Meyniel's Conjecture

For any graph G, $cop(G) = O(\sqrt{n})$.

Other Variants

- Cops and Attacking Robbers

Other Variants

- Cops and Attacking Robbers
- Lazy Cops and Robbers

Other Variants

- Cops and Attacking Robbers
- Lazy Cops and Robbers
- You guys come up with your own models!!!

Thank You!